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The edge covering number of the intersection of two matroids

Ron Aharoni [a,*], Eli Berger [b], Ran Ziv [c]

[a] Department of Mathematics, Technion, Haifa, 32000, Israel

[b] Department of Mathematics, Faculty of Natural Sciences, Haifa University, Haifa, 31905, Israel

[c] Department of Computer Science, Tel-Hai Academic College, Upper Galilee, 12210, Israel

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ABSTRACT

The *edge covering number* of a hypergraph $\mathcal{A}$ is

$$\beta(\mathcal{A}) = \min\left\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{A}, \bigcup \mathcal{B} = \bigcup \mathcal{A}\right\}.$$

The paper studies a conjecture on the edge covering number of the intersection of two matroids. For two natural numbers k, ℓ, let $f(k, \ell)$ be the maximal value of $\beta(\mathcal{M} \cap \mathcal{N})$ over all pairs of matroids $\mathcal{M}, \mathcal{N}$ such that $\beta(\mathcal{M}) = k$ and $\beta(\mathcal{N}) = \ell$. In (Aharoni and Berger, 2006) [1] the first two authors proved that $f(k, \ell) \leq 2\max(k, \ell)$ and conjectured that $f(k, k) = k + 1$ and $f(k, \ell) = \ell$ when $\ell > k$. In this paper we prove that $f(k, k) \geq k + 1$, $f(2, 2) = 3$ and $f(2, 3) \leq 4$. We also form a conjecture on the edge covering number of 2-polymatroids that is a common extension of the above conjecture and the Goldberg–Seymour conjecture, and prove its first non-trivial case.

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1. Introduction

The *edge covering number* of a hypergraph $\mathcal{A}$ is:

$$\beta(\mathcal{A}) = \min\left\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{A}, \bigcup \mathcal{B} = \bigcup \mathcal{A}\right\}.$$

Remark 1.1. The edge covering number is sometimes denoted by ρ, but this may be confused with matroid rank. In [1] the notation χ is used, the reason being that if G is a graph and $\mathcal{A}$ is the set of independent sets in G, then $\beta(\mathcal{A}) = \chi(G)$. But a graph is also a hypergraph, making the notation $\chi(G)$ ambiguous. Therefore we chose the β notation for the edge covering number, and r for matroid rank.

A *matroid* is a finite set of finite sets $\mathcal{M}$ satisfying:

- $\emptyset \in \mathcal{M}$,
- $X \in \mathcal{M}$ and $Y \subseteq X$ implies $Y \in \mathcal{M}$, and
- For every $X, Y \in \mathcal{M}$ with $|X| > |Y|$ there exists $x \in X \setminus Y$ for which $Y \cup \{x\} \in \mathcal{M}$.

The set $\bigcup \mathcal{M}$ is called the *ground set* of $\mathcal{M}$. In this paper we are interested in pairs of matroids $\mathcal{M}, \mathcal{N}$ on the same ground set and in estimating $\beta(\mathcal{M} \cap \mathcal{N})$ in terms of $\beta(\mathcal{M})$ and $\beta(\mathcal{N})$.

In [1] the first two authors proved:

* Corresponding author.
E-mail addresses: ra@tx.technion.ac.il (R. Aharoni), berger@math.haifa.ac.il (E. Berger), ranziv@telhai.ac.il (R. Ziv).

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Theorem 1.2. *If $\mathcal{M}$ and $\mathcal{N}$ are two matroids on the same ground set then $\beta(\mathcal{M} \cap \mathcal{N}) \leq 2\max(\beta(\mathcal{M}), \beta(\mathcal{N}))$.*

Example 1.3 (*[6, Section 42.6c]*)**.** Let $\mathcal{M}$ be the graphic matroid of K_4, i.e., the ground set of $\mathcal{M}$ is the edge set of K_4 and a set of edges belongs to $\mathcal{M}$ if and only if it does not contain the edge set of a cycle. Let $\mathcal{N}$ be the partition matroid on $E(K_4)$ in which the three parts are the three matchings of size 2 in K_4 (so, a subset of $E(K_4)$ belongs to $\mathcal{N}$ if every two edges in it meet).

In this example $\beta(\mathcal{M} \cap \mathcal{N}) = 3$, while $\beta(\mathcal{M}) = \beta(\mathcal{N}) = 2$. There is no example known at present in which the gap between $\beta(\mathcal{M} \cap \mathcal{N})$ and $\max(\beta(\mathcal{M}), \beta(\mathcal{N}))$ is larger than 1. Denoting by $f(k, \ell)$ the maximum of $\beta(\mathcal{M} \cap \mathcal{N})$ over all pairs of matroids $\mathcal{M}, \mathcal{N}$ such that $\beta(\mathcal{M}) = k$ and $\beta(\mathcal{N}) = \ell$, the following may well be true:

Conjecture 1.4 (*[1]*)**.** *For all $2 \leq k \leq \ell$ it is true that $f(k, \ell) = \max(k + 1, \ell)$.*

In the present paper we prove that $f(2, 2) = 3$ and $f(2, 3) \leq 4$. More explicitly:

Theorem 1.5. *If $\mathcal{M}$ and $\mathcal{N}$ are two matroids on the same ground set with $\beta(\mathcal{M}) = \beta(\mathcal{N}) = 2$, then $\beta(\mathcal{M} \cap \mathcal{N}) \leq 3$.*

Theorem 1.6. *If $\mathcal{M}$ and $\mathcal{N}$ are two matroids on the same ground set with $\beta(\mathcal{M}) = 2$ and $\beta(\mathcal{N}) = 3$, then $\beta(\mathcal{M} \cap \mathcal{N}) \leq 4$.*

We also prove the lower bound:

Theorem 1.7. *For every integer $k > 1$ there exist two matroids $\mathcal{M}, \mathcal{N}$ on the same ground set with $\beta(\mathcal{M}) = \beta(\mathcal{N}) = k$ and $\beta(\mathcal{M} \cap \mathcal{N}) = k + 1$.*

2. Proofs

Definition 2.1. Let $\mathcal{M}$ be a matroid. The *rank* $r(\mathcal{M})$ of $\mathcal{M}$ is $\max_{A \in \mathcal{M}} |A|$. A set $B \in \mathcal{M}$ of size $r(\mathcal{M})$ is called a *base* of $\mathcal{M}$.

Our main tool is the following well known "basis exchange property" (proved independently in [2,4]):

Lemma 2.2. *Let $\mathcal{M}$ be a matroid, let B, B' be two bases of $\mathcal{M}$ and let $X \subseteq B$. Then there exists a set $X' \subseteq B'$ such that $|X'| = |X|$ and both sets $(B \setminus X) \cup X'$ and $(B' \setminus X') \cup X$ belong to $\mathcal{M}$.*

The following lemma is not essential to the proof of the theorems, but it makes it more transparent. It is a reduction to the case in which the ground set can be partitioned into disjoint bases, in each of the two matroids:

Lemma 2.3. *For every two natural numbers k, ℓ there exist two matroids $\mathcal{M}, \mathcal{N}$ on the same ground set V such that $\beta(\mathcal{M}) = k$, $\beta(\mathcal{N}) = \ell$, $\beta(\mathcal{M} \cap \mathcal{N}) = f(k, \ell)$ and $|V| = kr(\mathcal{M}) = \ell r(\mathcal{N})$.*

Proof. Let $\mathcal{M}_0, \mathcal{N}_0$ be two matroids on the same ground set V_0 satisfying $\beta(\mathcal{M}_0) = k$, $\beta(\mathcal{N}_0) = \ell$ and $\beta(\mathcal{M}_0 \cap \mathcal{N}_0) = f(k, \ell)$. Write $n = |V_0|$. Let V be a set of size $k\ell n$ containing V_0 and write $\mathcal{M} = \{A \subset V : |A| \leq \ell n, A \cap V_0 \in \mathcal{M}_0\}$, $\mathcal{N} = \{A \subset V : |A| \leq kn, A \cap V_0 \in \mathcal{N}_0\}$. It is easy to check that $\mathcal{M}$ and $\mathcal{N}$ are matroids satisfying the requirements of the lemma. $\square$

Let $k \leq \ell$ be two natural numbers. Given a system of bases $B_1, B_2, \ldots, B_k$ of $\mathcal{M}$ and another system of bases $B^1, B^2, \ldots, B^\ell$ of $\mathcal{N}$, write B_i^j for $B_i \cap B^j$. The sum $\sum_{i=1}^{k} |B_i^i|$ is named the *trace* of the two systems.

Theorems 1.5 and 1.6 both follow from:

Theorem 2.4. *For every two natural numbers $k \leq \ell$ there exist two matroids $\mathcal{M}, \mathcal{N}$ on the same ground set V, satisfying the conditions of Lemma 2.3 and for which there exist a partition of V into bases $B_1, \ldots, B_k$ of $\mathcal{M}$ and a partition of V into bases $B^1, \ldots, B^\ell$ of $\mathcal{N}$ such that:*

(*) *For every three integers h, i, j with $h, i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, \ell\}$ and $h \neq i \neq j$ (but possibly $h = j$), we have $B_i^j \cup B_h^i \in \mathcal{M} \cap \mathcal{N}$.*

Proof. Let $\mathcal{M}, \mathcal{N}$ be as in Lemma 2.3. Choose two systems of bases $B_1, \ldots, B_k \in \mathcal{M}$ and $B^1, \ldots, B^\ell \in \mathcal{N}$ partitioning V, with maximal trace. Let i, j, h be as in the statement of the theorem. Apply Lemma 2.2 with $B = B_i$, $B' = B_h$ and $X = B_i^j$. Let X' be the set whose existence is assured by the lemma, and write $B_i' = (B_i \setminus X) \cup X'$, $B_h' = (B_h \setminus X') \cup X$. In the system of bases of $\mathcal{M}$ replace B_i by B_i' and B_h by B_h'. The system of bases of $\mathcal{N}$ remains the same. Since $i \neq h$, moving X to B_h has not reduced the trace of the two systems, and by the maximality of the trace this means that moving X' to B_i does not enlarge the trace. Since every element of $X' \cap B_h^i$ becomes an element of B_i^i (the latter taken in the new systems), this means that $X' \cap B_h^i = \emptyset$. But this means that $X \cup B_h^i \subseteq B_h'$, implying that $X \cup B_h^i = B_i^j \cup B_h^i \in \mathcal{M}$.

To prove that $B_i^j \cup B_h^i \in \mathcal{N}$ apply Lemma 2.2 with $B = B^i$, $B' = B^j$ and $X = B_h^i$. The argument is symmetrical to the one above. $\square$

Proof of Theorems 1.5 and 1.6. Let us first prove that $f(2,2)=3$. Take two matroids $\mathcal{M}, \mathcal{N}$ attaining $f(2,2)$, together with systems of bases satisfying the conclusion of Theorem 2.4. Taking $i=1, j=h=2$ in the theorem yields that $B_1^2 \cup B_2^1 \in \mathcal{M} \cap \mathcal{N}$. Then B_1^1, B_2^2 and $B_1^2 \cup B_2^1$ form a partition of V into three sets belonging to $\mathcal{M} \cap \mathcal{N}$, showing that $f(2,2) \le 3$. To prove that $f(2,3) \le 4$ note that for matroids and bases satisfying Theorem 2.4 the sets $B_1^1, B_2^2, B_2^1 \cup B_1^3$ and $B_1^2 \cup B_2^3$ form a partition of V into four sets belonging to $\mathcal{M} \cap \mathcal{N}$. □

We end this section with a proof of Theorem 1.7. The example is a straightforward generalization of Example 1.3, obtained by adding parallel edges. Let $G=(V,E)$ be the following multigraph. The vertex set is $V=\{1,2,3,4\}$, each of the sets $\{1,2\},\{1,3\},\{2,3\}$ appears once as an edge and each of the sets $\{1,4\},\{2,4\},\{3,4\}$ appears as an edge $k-1$ times. Denote the $k-1$ copies of the edge $\{i,4\}$ (for $i=1,2,3$) by $\{i,4\}_1,\ldots,\{i,4\}_{k-1}$. Define two matroids $\mathcal{M}, \mathcal{N}$ on the ground set E, as follows. For every $X \subseteq E$ let $X \in \mathcal{M}$ if and only if X contains neither two copies of the same edge nor a set of edges forming a cycle. Let $X \in \mathcal{N}$ if and only if X contains neither two copies of the same edge, nor two disjoint edges. Note that $X \in \mathcal{M} \cap \mathcal{N}$ if and only if X is a star.

Clearly $r(\mathcal{M}) = r(\mathcal{N}) = r(\mathcal{M} \cap \mathcal{N}) = 3$, and since $|E| = 3k$ it follows that $\beta(\mathcal{M}), \beta(\mathcal{N}), \beta(\mathcal{M} \cap \mathcal{N}) \ge k$. Writing $S_i = \{\{1,4\}_i, \{2,4\}_i, \{3,4\}_i\}$ for $i=1,\ldots,k-1$, the set $\{\{1,2\},\{2,3\},\{3,4\}_1\}$, the set $\{\{1,3\},\{1,4\}_1,\{2,4\}_1\}$, and the sets $S_2,\ldots,S_{k-1}$ form a partition of the ground set E, showing that $\beta(\mathcal{M}) = k$. Similarly, the partition $\{\{\{1,2\},\{2,3\},\{1,3\}\}, S_1,\ldots,S_{k-1}\}$ of E shows that $\beta(\mathcal{N}) = k$ and the partition $\{\{\{1,2\}\},\{\{2,3\},\{1,3\}\},S_1,\ldots,S_{k-1}\}$ shows that $\beta(\mathcal{M} \cap \mathcal{N}) \le k+1$. In order to see that $\beta(\mathcal{M} \cap \mathcal{N}) = k+1$ assume for contradiction that there exist $X_1,\ldots,X_k \in \mathcal{M} \cap \mathcal{N}$ such that $X_1 \cup \cdots \cup X_k = E$. Then the sets $X_1,\ldots,X_k$ are stars and by a counting argument each of them is of size 3 and every edge appears in exactly one of them. If two of the stars are centered at vertices from $\{1,2,3\}$, say 1 and 2, then the edge $\{1,2\}$ appears in two of the sets X_i, which yields a contradiction. If at most one of the stars is centered at a vertex from $\{1,2,3\}$, say 1, then the edge $\{2,3\}$ does not appear in any of the sets X_i, again yielding a contradiction.

3. The edge covering number of a 2-polymatroid

Definition 3.1. Let $\mathcal{A}$ be a hypergraph. A function $w : \mathcal{A} \to [0,1]$ is a *fractional edge cover* if for every $v \in \bigcup \mathcal{A}$ it is true that:

$$\sum_{A: v \in A \in \mathcal{A}} w(A) \ge 1.$$

The *fractional edge covering number* $\beta^*(\mathcal{A})$ of $\mathcal{A}$ is the minimal value of $\sum_{A \in \mathcal{A}} w(A)$ over all fractional edge covers of $\mathcal{A}$.

It is well known (see, e.g., [1]) that $\beta(\mathcal{M}) = \lceil \beta^*(\mathcal{M}) \rceil$ for any matroid $\mathcal{M}$ and that for two matroids $\mathcal{M}$ and $\mathcal{N}$ we have $\beta^*(\mathcal{M} \cap \mathcal{N}) = \max(\beta^*(\mathcal{M}), \beta^*(\mathcal{N}))$. Hence Conjecture 1.4 can be reformulated as follows:

Conjecture 3.2. *For two matroids $\mathcal{M}$ and $\mathcal{N}$ on the same ground set*

$$\beta(\mathcal{M} \cap \mathcal{N}) \le \beta^*(\mathcal{M} \cap \mathcal{N}) + 1.$$

This is reminiscent of a famous conjecture of Goldberg [3] and Seymour [7]:

Conjecture 3.3. *Let G be a multigraph, and write $M(G)$ for the hypergraph of matchings in G. Then*

$$\beta(M(G)) \le \beta^*(M(G)) + 1.$$

In this section we introduce a common generalization of the two conjectures, formulated in the terminology of 2-polymatroids. For the purposes of this paper 2-polymatroids are defined as follows:

Definition 3.4. Given a matroid $\mathcal{M}$ on the ground set $V \times \{1,2\}$, the hypergraph $\mathcal{P} = \{A \subseteq V : A \times \{1,2\} \in \mathcal{M}\}$ is called a 2-polymatroid.

The matroid $\mathcal{M}$ is called a *representation* of $\mathcal{P}$. (Note that a 2-polymatroid may have more than one representation.) More commonly, a k-polymatroid $\mathcal{P}$ is defined as a pair (V, r), where V is a set of vertices and $r = r_{\mathcal{P}}$ is a rank function from the power set of V to the set of nonnegative integers, which satisfies the same axioms as the rank function of a matroid, namely monotonicity and submodularity ($r(A \cup B) + r(A \cap B) \le r(A) + r(B)$). The difference from matroids is that the condition $0 \le r(\{a\}) \le 1$ (where $\{a\}$ is a singleton) is replaced by the condition $0 \le r(\{a\}) \le k$. The hypergraph associated with the k-polymatroid is then the set of sets X for which $r(X) = k|X|$. We shall restrict our attention only to this hypergraph, and with a slight abuse of notation use the name "k-polymatroid" for it. The link between the two definitions is given by a result of Lovász. Recall that a set of vertices in a matroid is called a *flat* if $r(F \cup \{x\}) > r(F)$ for every $x \notin F$. The rank $r(F)$ is called the *dimension* of F.

Lemma 3.5 ([5, Proposition 3.1])**.** *Every rank-defined k-polymatroid $P = (V, r_{\mathcal{P}})$ can be represented in such a way that V is a the set of k-dimensional flats of a certain matroid $\mathcal{M}$, and $r_{\mathcal{P}}$ is given by $r_{\mathcal{P}}(S) = r_{\mathcal{M}}\left(\bigcup S\right)$ for every set S of flats.*

Given a rank-defined 2-polymatroid $\mathcal{P}$ and a matroid $\mathcal{M}$ representing it as in the lemma, choose a pair of independent elements a, b from the 2-dimensional flat v for every v, and write $(v, 1)$ for a and $(v, 2)$ for b. It is easy to see that with this choice the hypergraph associated with the 2-polymatroid is precisely the one appearing in Definition 3.4.

Two of the best known examples of 2-polymatroids are:

(i) The intersection of two matroids $\mathcal{M}$ and $\mathcal{N}$, the rank function defining the 2-polymatroid being $r(X) = r_{\mathcal{M}}(X) + r_{\mathcal{N}}(X)$. Clearly, $r(X) = 2|X|$ if and only if $X \in \mathcal{M} \cap \mathcal{N}$.
(ii) The set of matchings of a multigraph, the rank function of a set of edges being $r(X) = \left|\bigcup X\right|$ (namely, the number of vertices participating in edges from X). Clearly, $\left|\bigcup X\right| = 2|X|$ if and only if X is a matching.

And here are the representations in the hypergraph terminology:

(i) A representation of the intersection of two matroids:
Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same ground set V. Let $\mathcal{M} \oplus \mathcal{N} = \{(A \times \{1\}) \cup (B \times \{2\}) : A \in \mathcal{M}, B \in \mathcal{N}\}$. Then $\mathcal{M} \oplus \mathcal{N}$ is a matroid and $\mathcal{M} \cap \mathcal{N} = \{A : A \times \{1, 2\} \in \mathcal{M} \oplus \mathcal{N}\}$.
(ii) A representation of the set of matchings in a multigraph:
Let $G = (V, E)$ be a multigraph and apply to its edges an arbitrary orientation. Let $\pi : E \times \{1, 2\} \to V$ be defined by $\pi(((u, v), 1)) = u$ and $\pi(((u, v), 2)) = v$. Let $\mathcal{M}$ be the set of all subsets of $E \times \{1, 2\}$ on which the restriction of the function π is one-to-one. It is easy to check that $\mathcal{M}$ is a matroid and that the set $\mathcal{P} = \{A : A \times \{1, 2\} \in \mathcal{M}\}$ is the set of all matchings in the multigraph G.
(To see this, note that a pair of edges $\{(u, v), (p, q)\}$ belongs to $\mathcal{P}$ if π is injective on $\{((u, v), 1), ((u, v), 2)((p, q), 1), ((p, q), 2)\}$, which means that all four vertices u, v, p, q are distinct.)

Conjectures 3.2 and 3.3 are thus both of the same type: claiming that certain 2-polymatroids satisfy $\beta \le \beta^* + 1$. As we shall see, this is not true for all 2-polymatroids, but it may be true for a certain type of 2-polymatroids, to which both the above examples belong.

Theorem 3.6. *In a k-polymatroid $\beta \le k\beta^*$.*

The proof follows a similar outline to that of the proof of Theorem 1.2 given in [1] (there it appears as Theorem 8.9). The proof is topological, and since we do not want to develop here the entire necessary topological machinery, we give here only a sketch. The *(homotopic) topological connectivity* $\eta(X)$ of a topological space X is the minimal dimension of a "hole" in X, namely

$$\eta(X) = \text{the minimal } d \text{ for which there exists a continuous function from } S^{d-1} \text{ into } X \text{ not extendibles to a continuous function from } B^d.$$

The connectivity of a simplicial complex (closed down hypergraph) is defined as the connectivity of its geometric realization in $\mathbb{R}^n$. The theorem follows from two facts:

Theorem 3.7. *For every simplicial complex H we have: $\beta(H) \le \max_{S \subseteq V(H)} \frac{|S|}{\eta(H|S)}$.*

Lemma 3.8. *If $\mathcal{P}$ is a k-polymatroid then $\eta(\mathcal{P}) \ge \frac{r(\mathcal{P})}{k}$.*

This property of k-polymatroids is proved in Theorem 6.5 in [1] for the intersection of k matroids. The proof for a general k-polymatroid $\mathcal{P}$ is very similar and uses the following combinatorial property of k-polymatroids: If $X \in \mathcal{P}$ with $|X| > k$ and $y \in \bigcup \mathcal{P}$ then $Z \cup \{y\} \in \mathcal{P}$ for some $Z \subseteq X$ of size $|Z| = |X| - k$.

We do not know whether the ratio 2 between β and β^* is attained in 2-polymatroids, but it can be bounded away from 1:

Theorem 3.9. *For every natural number k, there exists a 2-polymatroid $\mathcal{P}$ with $\beta(\mathcal{P}) = 2k$ and $\beta^*(\mathcal{P}) = \frac{3}{2}k$.*

Proof. For any graph $G = (V, E)$ we construct a corresponding polymatroid as follows, and write $V' = V \times \{1, 2\}$. Let $\mathcal{M}$ be the set of all subsets of V' of size at most 4, except for the sets of the form $\{(u, 1), (u, 2), (v, 1), (v, 2)\}$ where u and v are two distinct vertices and $\{u, v\} \notin E$. Is it easy to check that $\mathcal{M}$ is a matroid and therefore the set $\mathcal{P} = \{A : A \times \{1, 2\} \in \mathcal{M}\}$ is a 2-polymatroid.

It is not hard to see that in fact $\mathcal{P} = E \cup \{\{v\} : v \in V\} \cup \{\emptyset\}$. Setting G to be the graph consisting of k vertex disjoint triangles, we get $\beta(\mathcal{P}) = 2k$ and $\beta^*(\mathcal{P}) = \frac{3}{2}k$. $\square$

The theorem shows that the inequality $\beta \le \beta^* + 1$ does not hold for 2-polymatroids in general, but it may be true for a special class of 2-polymatroid. Recall that a *circuit* in a matroid is a minimal set not belonging to the matroid.

Definition 3.10. Let V be a set and let $C \subseteq V \times \{1, 2\}$. We say that C is *pair-shunning* if for every $v \in V$ we have $|C \cap \{(v, 1), (v, 2)\}| \le 1$.

Definition 3.11. Let $\mathcal{P}$ be a 2-polymatroid and let $\mathcal{M}$ be a representation of $\mathcal{P}$. We say that $\mathcal{M}$ is a *pair-shunning circuits representation* (or PSCR) of $\mathcal{P}$ if every circuit of $\mathcal{M}$ is pair-shunning.

Note that the representations above of the intersection of two matroids and of the set of matchings in a multigraph both have pair-shunning circuits. In the first example, a circuit is just $C \times \{1\}$ for a circuit C of $\mathcal{M}$, or $D \times \{2\}$ for a circuit D of $\mathcal{N}$. In the second example, all circuits are of the form $\{(e, i), (e', j)\}$ where $\pi((e, i)) = \pi((e', j))$. If the multigraph is loopless (which we can always assume, since loops do not appear in any matching) then $e \neq e'$, so the circuit is pair-shunning.

Conjecture 3.12. *If a 2-polymatroid $\mathcal{P}$ has a PSCR then $\beta(\mathcal{P}) \leq \beta^*(\mathcal{P}) + 1$.*

We prove the first non-trivial case of this conjecture:

Theorem 3.13. *If a 2-polymatroid $\mathcal{P}$ has a PSCR and $\beta^*(\mathcal{P}) \leq 2$ then $\beta(\mathcal{P}) \leq 3$.*

Proof. Write $V = \bigcup \mathcal{P}$ and $|V| = n$. For every $W \subseteq V \times \{1, 2\}$ let us write $\phi(W) = \{v \in V : (v, 1), (v, 2) \in W\}$. Let $\mathcal{M}$ be a PSCR of $\mathcal{P}$. It is easy to see that $\beta^*(\mathcal{M}) \leq \beta^*(\mathcal{P}) \leq 2$ and therefore $\beta(\mathcal{M}) \leq 2$. Standard methods of Matroid Theory show that the matroid $\mathcal{M}_n = \{A \in \mathcal{M} : |A| \leq n\}$ satisfies $\beta(\mathcal{M}_n) = 2$. Choose bases $B_1, B_2 \in \mathcal{M}_n$ satisfying $B_1 \cup B_2 = V \times \{1, 2\}$ in a way that $|\phi(B_1)| + |\phi(B_2)|$ is as large as possible. Write $V_1 = \phi(B_1)$, $V_2 = \phi(B_2)$, $V_3 = V \setminus (V_1 \cup V_2)$. Clearly $V_1, V_2 \in \mathcal{P}$. We also claim that $V_3 \in \mathcal{P}$ and hence $\beta(\mathcal{P}) \leq 3$. Assume for contradiction that $V_3 \notin \mathcal{P}$. In other words, the set $W = V_3 \times \{1, 2\}$ is not in $\mathcal{M}$. Therefore W contains some circuit C of $\mathcal{M}$. By Lemma 2.2 with $X = B_1 \cap C$, there exists a set $X' \subseteq B_2$ satisfying $|X'| = |X|$ and $(B_1 \setminus X) \cup X'$, $(B_2 \setminus X') \cup X \in \mathcal{M}_n$. Write $B_1' = (B_1 \setminus X) \cup X'$, $B_2' = (B_2 \setminus X') \cup X$. Note that $\phi(B_1') \supseteq \phi(B_1)$ and $|\phi(B_2')| \geq |\phi(B_2)| + |X| - |X'| = |\phi(B_2)|$. By the maximality of $|\phi(B_1)| + |\phi(B_2)|$ we must have $|\phi(B_2')| = |\phi(B_2)|$ and this can only happen if $X' \subseteq V_2 \times \{1, 2\}$. This implies $X' \cap W = \emptyset = X' \cap C$, and $B_2 \cap C \subseteq B_2 \setminus X'$. Therefore

$$C = (C \cap B_2) \cup (C \cap B_1) \subseteq (B_2 \setminus X') \cup X = B_2' \in \mathcal{M}$$

which is a contradiction to the assumption that C is a circuit of $\mathcal{M}$. $\square$

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